

GENERALIZATION OF AN INEQUALITY OF BIRNBAUM AND MARSHALL, WITH APPLICATIONS TO GROWTH RATES FOR SUBMARTINGALES

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The well-known submartingale maximal inequality of Birnbaum and Marshall (1961) is generalized to provide upper tail inequalities for suprema of processes which are products of a submartingale by a nonincreasing nonnegative predictable process. The new inequalities are proved by applying an inequality of Lenglart (1977), and are then used to provide best-possible universal growth-rates for a general submartingale in terms of the predictable compensator of its positive part. Applications of these growth rates include strong asymptotic upper bounds on solutions to certain stochastic differential equations, and strong asymptotic lower bounds on Brownian-motion occupation-times.

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continuous-time submartingale * predictable compensator * Lenglart Inequality * strong asymptotic growth rate * occupation times for Brownian Motion

1. Introduction

A large part of the usefulness of submartingales as a class of stochastic processes derives from the availability of upper-tail probability inequalities like Doob's (Doob, 1953, Theorem 3.4; Liptser and Shiryaev, 1977, vol. 1, Theorem 3.2) for their suprema. Such inequalities are of great importance in many key calculations of probability theory, e.g. in establishing asymptotic rates of growth for proving various sorts of strong laws, in checking tightness of sequences of laws of stochastic processes, and generally in relating the stochastic behavior of the supremum of a process on an interval to its stochastic behavior at the right endpoint of the interval.

The inequality of Birnbaum and Marshall (1961, Theorem 5.1) which we generalize in this paper is a very useful extension of Doob's inequality because it gives a tail-probability bound for the supremum of a process which need not itself be a submartingale [so that Doob's maximal inequality can not be applied directly] but which is a product of a submartingale by a nonnegative, nonincreasing, right-continuous, adapted function. The Birnbaum-Marshall inequality includes the

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Chebychev, Kolmogoroff, and Hájek–Rényi inequalities [Loève, 1955, pp. 235, 386; Petrov, 1975, p. 51], which are among the handiest inequalities concerning the maxima of partial sums of independent summands. The power of the (generalized) Birnbaum–Marshall inequality can be seen clearly in the uses to which it will be put in Sections 3 and 4.

The new observation motivating the present work is that the inequality of Birnbaum and Marshall can be proved and improved by means of the following inequality due to Lengart (1977). This inequality of Lengart can be regarded as the “master inequality” of the Doob type for tail probabilities of suprema of processes which can be “dominated” by a right-continuous increasing predictable process.

Theorem 1.1 [Lengart inequality, 1977]. *If $X(\cdot)$ is a $\{\mathcal{F}(t)\}$ -adapted process [on a probability space (Ω, \mathcal{F}, P)] which is a.s. in $D[0, \infty)$ as a random function and which satisfies $X(0) = 0$ a.s., and if $A(\cdot)$ is any right-continuous nondecreasing $\{\mathcal{F}(t)\}$ -predictable process such that (i) $A(0) = 0$, and (ii) for any bounded stopping-time τ , $\mathcal{E}X(\tau) \leq \mathcal{E}A(\tau)$, then for arbitrary positive constants c and d and every essentially bounded stopping-time τ ,*

$$P\left\{\sup_{0 \leq s \leq \tau} X(s) \geq c\right\} \leq c \cdot \mathcal{E}\{A(\tau)^+ d\} + P\{A(\tau) \geq d\}.$$

Note. Here and throughout the paper, if $I = [a, b]$ or $[a, b)$ is a subinterval of $[0, \infty)$, then $D(I)$ denotes the collection of right-continuous real-valued functions on I which have limits from the left at all interior points of I . In addition, the notation $x \wedge y$ denotes $\min\{x, y\}$. Finally, we assume throughout the paper that the nondecreasing family $\{\mathcal{F}(t): t \geq 0\}$ of sub- σ -fields of \mathcal{F} on Ω is right-continuous in the sense that $\mathcal{F}(t+) \equiv \bigcap_{s > t} \mathcal{F}(s) = \mathcal{F}(t)$.

To show the idea of our inequality without technicalities related to continuous time, we first re-prove the inequality of Chow (1960) which is the discrete-time specialization of our Theorem 2.1.

Proposition 1.2 [Chow, 1960]. *Let $\{Y_k\}_{k \geq 0}$ with $Y_0 = 0$ be any $\{\mathcal{F}_k\}$ -adapted sequence of integrable r.v.'s on a fixed probability space (Ω, \mathcal{F}, P) , where $\{\mathcal{F}_k\}_{k \geq 0}$ is an increasing family of sub- σ -fields of \mathcal{F} . In addition, let $\{a_k\}_{k \geq 0}$ with $a_0 > 0$ be any a.s. nonincreasing positive sequence of random variables with each a_k measurable with respect to \mathcal{F}_{k-1} . Then*

$$P\left\{\sup_k Y_k a_k \geq 1\right\} \leq \mathcal{E} \sum_k a_{k-1} (B_k - B_{k-1}),$$

where $\{B_k\}$ is defined by $B_k \equiv \sum_{j=1}^k \mathcal{E}\{Y_j^+ - Y_{j-1}^+ | \mathcal{F}_{j-1}\}$.

Proof. We will apply a discrete-time specialization of Theorem 1.1 (which is very easy to prove directly in the discrete-time setting) to the sequence $X_k \equiv Y_k^+ a_k$ (where

$x^+ \equiv \max\{x, 0\}$ and the dominating sequence $A_k \equiv \sum_{j=1}^k a_{j-1}(B_j - B_{j-1})$. We first calculate for any bounded stopping-time $\tau \leq N$,

$$\begin{aligned} \mathcal{E}X_\tau &\equiv \mathcal{E}Y_\tau^+ a_\tau = \mathcal{E} \sum_{i=0}^{N-1} (Y_{(i+1)^+}^+ \cdot a_{(i+1)^+} - Y_{i^+}^+ a_{i^+}) \\ &\leq \sum_{i=0}^{N-1} \mathcal{E}I_{[\tau > i]} a_i (Y_{(i+1)^+}^+ - Y_{i^+}^+) \\ &= \sum_{i=0}^{N-1} \mathcal{E}I_{[\tau > i]} a_i (B_{i+1} - B_i) = \mathcal{E} \sum_{j=1}^{\tau} a_{j-1} (B_j - B_{j-1}) \equiv \mathcal{E}A_\tau. \end{aligned}$$

Regarding the processes X_t and A_t as constant on each time-interval $t \in [k, k+1)$, we see that these processes satisfy the hypotheses of Theorem 1.1. Therefore

$$P\{\sup_k Y_k^+ a_k \geq c\} \leq c^{-1} \mathcal{E} \sum_{j=1}^N a_{j-1} (B_j - B_{j-1}).$$

Finally, let $N \rightarrow \infty$ and apply the Monotone Convergence Theorem to complete the proof. \square

The plan of the paper is as follows. Section 2 contains the statement and proof of the generalized inequality of the title. Section 3 applies it to give an upper-class type result for any submartingale in terms of the predictable compensator of its positive part as the time-parameter goes to ∞ . The resulting growth-estimate is shown by an example to be best-possible. Finally, Section 4 contains two quite different applications of the growth-estimate, one to a class of stochastic differential equations and one to the asymptotics of Brownian-motion occupation-times for complicated subsets of the real line.

2. Generalized Birnbaum–Marshall inequality

Theorem 2.1. *Suppose that $X(\cdot)$ is a locally square-integrable $\{\mathcal{F}(t): t \geq 0\}$ -submartingale which is a.s. in $D[0, T]$ [where $T \leq \infty$] and for which $X(0) = 0$ a.s. Suppose also that $h(\cdot)$ is a predictable $\{\mathcal{F}(t)\}$ -adapted random function which is a.s. nonincreasing and nonnegative. Then*

$$P\{\sup_{0 \leq s < T} X(t) \cdot h(t+) \geq 1\} \leq \mathcal{E} \int_0^T h(s) dA_+(s),$$

where $A_+(\cdot)$ is the predictable compensator of $X^+(\cdot) \equiv \max\{X^+(\cdot), 0\}$. In particular, if $h(\cdot)$ is nonrandom, then

$$P\{\sup_{0 \leq s < T} X(t) \cdot h(t+) \geq 1\} \leq \int_0^T h(s) d\mu(s),$$

where $\mu(s) \equiv \mathcal{E}X^+(s)$ for $s \in [0, T]$.

Proof. Since $\max\{x, 0\}$ is a nondecreasing convex function of x , $X^+(\cdot)$ is a (locally square-integrable) $\{\mathcal{F}(t)\}$ -submartingale. The general Doob-Meyer Decomposition theorem implies the existence of the compensator $A_+(\cdot)$. Let $C > 0$ be arbitrary, and let $\{\tau_n\}$ be a sequence of stopping times increasing a.s. to T such that $\mathcal{E}X^2(\tau_n) < \infty$ for each n . Our main task is to apply Lengart's inequality to the right-continuous process

$$Y(s) \equiv X^+(s \wedge \tau_n) \cdot \min\{C, h_+(s \wedge \tau_n)\}, \quad 0 \leq s < T,$$

for each n [where we make the notational definition $h_+(s) \equiv h(s+)$] along with the predictable process

$$A(\cdot) \equiv \int_0^{\tau_n} \min\{h(s), C\} dA_+(s).$$

Clearly $Y(\cdot)$ is a.s. an element of $D[0, T]$ satisfying $Y(0) = 0$, and $A(\cdot)$ is a.s. right-continuous with $A(0) = 0$. We shall verify for all stopping-times $\tau \leq T$, that

$$\mathcal{E}Y(\tau) \leq \mathcal{E}A(\tau). \quad (2.1)$$

This can be done directly using results from stochastic calculus. Indeed, if we denote $h_c(\cdot) \equiv \min\{h(\cdot), C\}$, and if we define $M(\cdot) \equiv M_n(\cdot)$ to be the square-integrable martingale $X^+(\cdot \wedge \tau_n) - A_+(\cdot \wedge \tau_n)$, then Theorem (2.53)(b) of Jacod (1979) implies that

$$\begin{aligned} X^+(t \wedge \tau_n) h_c(t \wedge \tau_n) &= \int_0^{t \wedge \tau_n} X^+(s-) dh_c(s) + \int_0^{t \wedge \tau_n} h_c(s) dX^+(s) \\ &= \int_0^{t \wedge \tau_n} X^+(s-) dh_c(s) + \int_0^{t \wedge \tau_n} h_c(s) dM(s) \\ &\quad + \int_0^{t \wedge \tau_n} h_c(s) dA_+(s). \end{aligned}$$

[It is worth remarking here that the result we have used from stochastic calculus amounts to nothing more than integration by parts in case $h(\cdot)$ is left-continuous.] In the last expression, the first integral is a.s. ≤ 0 because $h(\cdot)$ is nonincreasing, and the second integral is a square-integrable martingale. Therefore, replacing t by an arbitrary stopping-time τ , taking expectations, and relying on the Optional Sampling Theorem to ensure that the expectation of the middle integral-term is 0, we find

$$\mathcal{E}Y(\tau) \leq \mathcal{E}\{X^+(\tau_n \wedge \tau) h_c(\tau_n \wedge \tau)\} \leq \mathcal{E} \int_0^{\tau \wedge \tau_n} h_c(s) dA_+(s) \equiv \mathcal{E}A(\tau)$$

as was to be shown.

Now Lengart's Inequality applied to the processes $Y(\cdot)$ and $A(\cdot)$ for fixed n and C says in particular [for $c = 1$ and $d = \infty$ in Theorem 1.1]

$$P\left\{\sup_{0 \leq s \leq \tau_n} X^+(s) h_c(s+) \geq 1\right\} \leq \mathcal{E} \int_0^{\tau_n} h_c(s) dA_+(s). \quad (2.2)$$

The proof of the Theorem is completed by letting $n \rightarrow \infty$ and $C \rightarrow \infty$ and applying the Monotone Convergence Theorem to both sides of (2.2).

If $h(\cdot)$ is nonrandom, then $\mathcal{E}A(T)$ is by the Fubini-Tonelli Theorem equal to $\int_0^T h(s) d\mu(s)$, and the Theorem is proved. \square

3. Universal rate-of-growth estimates for submartingales

It is well known that if the submartingale $X(\cdot) \equiv M^2(\cdot)$ is the square of a zero-mean martingale with almost-surely continuous paths, then the Law of the Iterated Logarithm for Brownian Motion has the consequence that a.s. on the event $\langle M \rangle(\infty) = \infty$,

$$\limsup_{t \rightarrow \infty} X(t) / \{2A(t) \log \log A(t)\} = 1 \quad (3.0)$$

where the compensator $A(\cdot)$ of $X(\cdot)$ is the same as the predictable variance $\langle M \rangle(\cdot)$ of $M(\cdot)$ [see Durrett, 1984, p. 77, where references are also given to analogous results for squares of discontinuous martingales]. On the other hand, there are no corresponding results known to the author on strong rates of growth in terms of $A(\cdot)$ for general submartingales $X(\cdot)$. Our main application of Theorem 2.1 is to prove such a result.

Theorem 3.1. *Suppose that $X(t)$ for $t \geq 0$ is a locally square-integrable $\{\mathcal{F}_t\}$ -submartingale with $X(0) = 0$ and $X(\cdot) \in D[0, T]$ a.s. for each $T < \infty$. Let $A_+(\cdot)$ denote the predictable compensator of $X^+(\cdot)$; define for each $m > 0$, $\tau(m) \equiv \inf\{t > 0: A_+(t) \geq m\}$, and let $q(t)$ be any nonrandom positive increasing function such that, for some $m > 0$,*

$$\mathcal{E} \int_{\tau(m)}^{\infty} [q(A_+(u))]^{-1} dA_+(u) < \infty. \quad (3.1)$$

Then on the event $[A_+(T) \rightarrow \infty \text{ as } T \rightarrow \infty]$, almost surely

$$\limsup_{T \rightarrow \infty} X(T) / q(A_+(T)) = 0.$$

Proof. Let $q(\cdot)$ be as in the hypotheses of the Theorem. Then the random function $h(t) \equiv 1/q(A_+(\max\{\tau(2), t\}))$ is predictable, right-continuous, and a.s. nonincreasing. Therefore Theorem 2.1 applied to $X^+(\max\{\tau(m), \cdot\}) - X^+(\tau(m))$, $m \geq 2$, says that

$$\begin{aligned} & P\{\sup\{[X^+(s) - X^+(\tau(m))] \cdot h(s): \tau(m) \leq s\} \geq c\} \\ & \leq c^{-1} \mathcal{E} \left\{ \int_{\tau(m)}^{\infty} h(s) dA_+(s) \right\} = c^{-1} \mathcal{E} \int_{\tau(m)}^{\infty} [q(A_+(s))]^{-1} dA_+(s). \end{aligned}$$

Now fix a nonrandom sequence $\{m(k): k = 1, 2, \dots\}$ of real numbers which increases to ∞ so rapidly that

$$\sum_{k=1}^{\infty} \mathcal{E} \int_{\tau(m(k))}^{\infty} [q(A_+(u))]^{-1} dA_+(u) < \infty.$$

[This can be done because the finiteness of the expression in (3.1) implies that the same expression must converge to 0 as $m \rightarrow \infty$.] The inequality of the previous paragraph together with the Borel-Cantelli Lemma imply for each $c > 0$ [since the right-hand sides of the inequality are summable in k when m is replaced by $m(k)$] that

$$P\{\sup\{(X^+(s) - X^+(\tau(m(k)))) \cdot h(s) : \tau(m(k)) \geq s\} \geq c \text{ i.o. in } k\} = 0.$$

It follows that a.s. on the event $[A_+(T) \uparrow \infty \text{ as } T \uparrow \infty]$, for all $k \geq \text{some } k(\omega, c)$,

$$\sup\{[X^+(s) - X^+(\tau(m(k)))] \cdot h(s) : s \geq \tau(m)\} \leq c.$$

Thus for all $k \geq k_0(\omega, c)$ [for a set of ω with probability 1]

$$\sup_{\tau(m(k)) \leq s} X^+(s) \cdot h(s) \leq X^+(\tau(m(k))) \cdot h(\tau(m(k))) + c.$$

In other words, we have proved that a.s. when $A_+(\cdot)$ increases to ∞ ,

$$X^+(s) = O(q(A_+(s))) \text{ as } s \rightarrow \infty, \text{ i.e. } \limsup_{s \rightarrow \infty} X(s)/q(A_+(s)) < \infty.$$

To see that the last almost-sure \limsup is in fact 0, it suffices to remark [as is standard in the theory of integral tests for stochastic rates of growth] that whenever assumption (3.1) holds for a nonrandom nondecreasing function $q(\cdot)$, it must also hold for another such function $Q(\cdot)$ for which $q(t)/Q(t)$ increases sufficiently slowly to ∞ as $t \rightarrow \infty$. Then the conclusion that $X^+(s) = O(Q(A_+(s)))$ as s and $A_+(s) \rightarrow \infty$ immediately implies that

$$\limsup_{s \rightarrow \infty} X^+(s)/q(A_+(s)) = 0 \quad \text{a.s. on } [A_+(\infty) = \infty]. \quad \square$$

The hypothesis (3.1) takes a much simpler form in case the compensator $A_+(\cdot)$ is either continuous or has jumps which are strictly controlled in magnitude. As Example 3.3 will show, bounding the sizes of jumps of $A_+(\cdot)$ can be much less restrictive than bounding the jumps of $X(\cdot)$ itself. In what follows, the jump $f(t) - f(t-)$ at t of a function f in $D([0, T])$ is denoted $\Delta f(t)$.

Lemma 3.2. *If the submartingale $X(\cdot)$ is such that the predictable compensator A_+ of X^+ satisfies $\sup_t \Delta A_+(t) \leq C$ a.s. for a finite constant C , then the hypothesis (3.1) of Theorem 3.1 on the nondecreasing function $q(\cdot)$ is equivalent to the condition*

$$\int_m^x [q(x)]^{-1} dx < \infty \quad \text{for some finite } m > 0.$$

Proof. The hypothesis implies that $x \leq A_+(\tau(x)) \leq x + C$ for all x , so that $A_+(\tau(m + 2(k+1)C)) - A_+(\tau(m + 2kC))$ lies between C and $3C$, and

$$\int_{\tau(m)}^x [q(A_+(u))]^{-1} dA_+(u) = \sum_{k=0}^{\infty} \int_{\tau(m+2kC)}^{\tau(m+2(k+1)C)} [q(A_+(u))]^{-1} dA_+(u)$$

lies between

$$\frac{1}{2} \sum_{k=0}^{\infty} \int_{A_+(\tau(m+2kC))}^{A_+(\tau(m+2(k+1)C))} [q(x+3C)]^{-1} dx, \sum_{k=0}^{\infty} \int_{A_+(\tau(m+2kC))}^{A_+(\tau(m+2(k+1)C))} \frac{3 dx}{2q(x)}.$$

Thus

$$\frac{1}{2} \int_{m+4C}^{\infty} [q(x)]^{-1} dx \leq \int_{\tau(m)}^{\infty} [q(A_+(u))]^{-1} dA_+(u) \leq \int_m^{\infty} \frac{3 dx}{2q(x)},$$

and (3.1) is evidently equivalent to the integrability of $1/q$ on $[m, \infty)$ for sufficiently large m . \square

Example 3.3. Define a right-continuous nonnegative submartingale $X(t)$ to be constant on each interval $[j, j+1)$ for $j=0, 1, 2, \dots$, with $X(0) \equiv 0$, and

$$P\{X(j+1)=k|(X(s): s \leq j)\} = \begin{cases} (k-1)/k & \text{if } X(j)=k-1 > 0, \\ (n+1)^{-1} & \text{if } X(j)=n > 0, k=0, \\ 1 & \text{if } X(j)=0, k=1. \end{cases}$$

Then $\mathcal{E}\{X(j+1)-X(j)|(X(s): s \leq j)\} = I_{\{X(j)=0\}}$, so that $A_+(t) = \sum_{j \geq 0} I_{\{X(j) > 0, j \leq t-1\}}$. We note in passing that the jumps of $A_+(\cdot)$ have size at most 1, so that Lemma 3.2 applies, while there is no upper bound on the sizes of jumps of $X(\cdot)$.

Now let the sequence $\{\tau_j\}$ of times when $X(\cdot)$ returns to 0 be defined for $j \geq 0$ by $\tau_j \equiv \inf\{k \geq 0: A_+(k) > j\}$. Then it is easy to see that since $X(\cdot)$ is a Markov process, the sequence of waiting-times $T_i \equiv \tau_i - \tau_{i-1} - 1$ for $i \geq 1$ are independent and identically distributed random variables. An easy calculation shows that $P\{T_i \geq n\} = n^{-1}$ for $n \geq 1$. Since T_i is the largest value taken on by $X(\cdot)$ between times τ_{i-1} and τ_i , for any nondecreasing strictly positive function $q(\cdot)$

$$\limsup_{x \rightarrow \infty} (X(s)/q(A_+(s))) = \limsup_{k \rightarrow \infty} T_k/q(k).$$

But for any constant $b > 0$, by the Borel-Cantelli Lemmas

$$P\{T_k \geq bq(k) \text{ i.o. } (k)\} = \begin{cases} 1 & \text{if } \sum_n [bq(n)]^{-1} = \infty, \\ 0 & \text{if } \sum_n [bq(n)]^{-1} < \infty, \end{cases} \quad (3.2)$$

where $[x]$ again denotes the greatest integer $\leq x$, and the summations in (3.2) are taken over all positive integers n for which $b \cdot q(n) \geq 1$. Since b is arbitrary, and since the convergence or divergence of the sums in (3.2) is equivalent to the convergence or divergence of the integral

$$J \equiv \int_1^{\infty} [q(x)]^{-1} dx,$$

we conclude that in this example, almost surely

$$\limsup_{s \rightarrow \infty} X(s)/q(A_+(s)) = \begin{cases} \infty & \text{if } J = \infty, \\ 0 & \text{if } J < \infty. \end{cases}$$

Taking Corollary 3.2 into account, we note that Theorem 3.1 says in this Example precisely that the lim sup is zero a.s. if J is finite. Theorem 3.1 shows that $X(s) = o(q(A_+(s)))$ a.s. Thus our example shows that the rate-of-growth given by the Theorem is best-possible in the sense of identifying those functions q for which $X(s) = o(q(A_+(s)))$ a.s. on $[A_+(\infty) = \infty]$.

4. Applications to SDE's and Brownian occupation-times

4.1. Growth of solutions to stochastic differential equations

Consider first a solution $X(t)$ to the stochastic differential equation

$$dX(t) = g(t) dt + b(t) dW(t), \quad X(0) = 1,$$

for (possibly random, but progressively measurable and nonanticipating) functions b and g which are everywhere nonnegative. Then $X(\cdot)$ is a local submartingale which, under mild conditions on g and b , is a.s. path-continuous. Such a model arises naturally in Inventory Theory or in Economics where $g(\cdot)$ represents an income rate, measured in constant dollars (corrected for inflation over time from time-origin 0), and where $b(t) dW(t)$ represents the current fortune at time t multiplied by the instantaneous interest rate minus the instantaneous inflation rate.

Theorem 3.1 allows us to bound the order-of-magnitude of the nonnegative long-term fortune $X(t^{\wedge}\sigma_n)$ measured until the time $\sigma_n \equiv \inf\{s > 0; X(s) \leq 0\}$ of "bankruptcy" in terms of the compensator

$$A(t) \equiv \int_0^{t^{\wedge}\sigma_n} g(s) ds$$

for $X(\cdot^{\wedge}\sigma_n)$, on the event $[\sigma_n = \infty, A(\infty) = \infty]$ where $A(\cdot)$ increases a.s. to ∞ without reaching 0. The bound given by Theorem 3.1 is rather different from, and should be compared with, the iterated logarithm bound (3.0) on the submartingale

$$\left(X(t) - \int_0^t g(s) ds \right)^2 \quad \text{in terms of} \quad \int_0^t b^2(s) ds.$$

4.2. Total occupation times for Brownian motion

Now let $W(t)$ be a standard real-valued Brownian motion process, and let B be any (possibly unbounded) Lebesgue-measurable subset of the line. Define the

real-valued, twice-differentiable, nonnegative, increasing convex function $g(x)$ through the equation

$$g''(x) = 2I_B(x), \quad x \in R, \quad g(0) = 0, \quad (4.1)$$

Then $g(W(t))$ is by Itô's Lemma a submartingale with compensator $\int_0^t I_B(W_s) ds \equiv L_t(B)$ [with respect to $\sigma(W_u: 0 \leq u \leq t)$]. This compensator is otherwise known as the total occupation time for the Wiener process $W(\cdot)$ and the set B up to time t .

Theorem 3.1 implies therefore that

$$\limsup_{t \rightarrow \infty} g(W(t))/q(L_t(B)) = 0 \quad \text{a.s.}$$

for any positive increasing function q such that $1/q$ is Lebesgue integrable on $[0, \infty)$. In particular, since g is an increasing function for nonnegative values of its argument, and since the law of iterated logarithm (3.0) bounds the positive values of $W(t)$ for all sufficiently large t , we conclude

$$\liminf_{t \rightarrow \infty} q(L_t(B))/g([2t \log \log t]^{1/2}) = \infty \quad \text{a.s.} \quad (4.2)$$

When B is a bounded set, this kind of result should, in a spirit like that of Donsker and Varadhan (1977, p. 708, remarks following formula (1.4)) be regarded as a Strong Law constraining the possible growth of $X(t)$ as $t \rightarrow \infty$. [The reasoning is that asymptotic bounds below for $L_t(B)$ indirectly imply that $X(t)$ cannot grow too quickly.] In the cases where B has finite total Lebesgue measure $\lambda(B)$, it is easy to check that the function $g(x)$ defined by (4.1) behaves asymptotically for large x like $\lambda(B) \cdot x$, so that (4.2) says (for strictly increasing q) that

$$L_t(B)/q^{-1}([2t \log \log t]^{1/2}) \rightarrow \infty \quad \text{a.s. as } t \rightarrow \infty.$$

The laws of iterated logarithm of Kesten (1965) and Donsker and Varadhan (1977) for local times imply [for B with $\lambda(B) < \infty$] that

$$L_t(B) = O([t \log \log t]^{1/2}) \quad \text{a.s. as } t \rightarrow \infty.$$

Thus for B with finite total Lebesgue measure, our new result (4.2) complements known iterated logarithm results.

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